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# Invariant Submanifold of $\tilde{\psi}(3,2,1)$ Structure Manifold

**Abstract**

In this paper, we have studied various properties of a $\tilde{\psi}(3,2,1)$ structure manifold and its invariant submanifold. Under two different assumptions, the nature of induced structure $\psi$, has also been discussed.

**Keywords**: Invariant submanifold, Nijenhuis tensor, projection operators and complementary distributions.

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**Introduction**

Let $V^m$ be a $C^\infty$ m-dimensional Riemannian manifold imbedded in a $C^\infty$ n-dimensional Riemannian manifold $M^n$, where $m<n$. The imbedding being denoted by

$f : V^m \longrightarrow M^n$ Let B be the mapping induced by $f$ i.e. $B=df$

$df : T(V) \longrightarrow T(M)$

Let $T$ $(V,M)$ be the set of all vectors tangent to the submanifold $f(V)$. It is well known that $B : T(V) \longrightarrow T(V,M)$

Is an isomorphism. The set of all vectors normal to $f(V)$ forms a vector bundle over $f(V)$, which we shall denote by $N(V,M)$. We call $N(V,M)$ the normal bundle of $V^m$. The vector bundle induced by f from $N(V,M)$ is denoted by $N(V)$. We denote by $C : N(V) \longrightarrow N(V,M)$ the natural isomorphism and by $\eta_s^r(V)$ the space of all $C^\infty$ tensor fields of type $(r,s)$ associated with N (V). Thus $\zeta_0^0(V)=\eta_0^0(V)$ is the space of all $C^\infty$ functions defined on $V^m$ while an element of $\eta_0^1(V)$ is a $C^\infty$ vector field normal to $V^m$ and an element of $\zeta_0^1(V)$ is a $C^\infty$ vector field tangential to $V^m$.

Let $\bar{X}$ and $\bar{Y}$ be vector fields defined along $f(V)$ and $\tilde{X},\tilde{Y}$ be the local extensions of $\bar{X}$ and $\bar{Y}$ respectively. Then $[\tilde{X},\tilde{Y}]$ is a vector field tangential to $M^n$ and its restriction $[\tilde{X},\tilde{Y}]/f(V)$ to $f(V)$ is determined independently of the choice of these local extension $\tilde{X}$ and $\tilde{Y}$. Thus $[\bar{X},\bar{Y}]$ is defined as

$$[\bar{X},\bar{Y}]=[\tilde{X},\tilde{Y}]/f(V) \tag{1.1}$$

Since B is an isomorphism

$$[BX, BY]=B[X,Y] \text{ for all } X,Y \in \zeta_0^1(V) \tag{1.2}$$

Let $\bar{G}$ be the Riemannain metric tensor of $M^n$, we define $g$ and $g^*$ on $V^m$ and $N$ $(V)$ respectively as

$$g(X_1, X_2)=\tilde{G}(BX_1, BX_2)\, f, \text{ and} \tag{1.3}$$

$$g^*(N_1, N_2)=\tilde{G}(CN_1, CN_2) \tag{1.4}$$

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For all

$X_1, X_2 \in \zeta_0^1(V)$ and $N_1, N_2 \in \eta_0^1(V)$

It can be verified that $g$ and $g^*$ are the induced metrics on $V^m$ and $N(V)$ respectively.

Let $\tilde{\nabla}$ be the Riemannian connection determined by $\tilde{G}$ in $M^n$, then $\tilde{\nabla}$ induces a connection $\nabla$ in $f(V)$ defined by

$$\nabla_{\bar{X}}\bar{Y} = \tilde{\nabla}_{\tilde{X}}\tilde{Y} / f(V) \tag{1.5}$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary $C^\infty$ vector fields defined along $f(V)$ and tangential to $f(V)$.

Let us suppose that $M^n$ is a $C^\infty\,\tilde{\psi}(3,2,1)$ structure manifold with structure tensor $\tilde{\psi}$ of type (1,1) satisfying

$$\tilde{\psi}^3 + \tilde{\psi}^2 + \tilde{\psi} = 0 \tag{1.6}$$

Let $\tilde{L}$ and $\tilde{M}$ be the complementary distributions corresponding to the projection operators

$$\tilde{l} = \tilde{\psi}^3, \qquad \tilde{m} = I - \tilde{\psi}^3 \tag{1.7}$$

where I denotes the identity operator.

From (1.6) and (1.7), we have

(a) $\tilde{l} + \tilde{m} = I$ (b) $\tilde{l}^2 = \tilde{l}$

(c) $\tilde{m}^2 = \tilde{m}$

(d) $\tilde{l}\,\tilde{m} = \tilde{m}\,\tilde{l} = 0$ (1.8)

Let $D_l$ and $D_m$ be the subspaces inherited by complementary projection operators l and m respectively.

$$D_l = \{X \in T_p(V) :\ lX = X, mX = 0\}$$

$$D_m = \{X \in T_p(V) :\ mX = X, lX = 0\}$$

Thus $T_p(V) = D_l + D_m$

Also $Ker\ l = \{X : lX = 0\} = D_m$

$Ker\ m = \{X : mX = 0\} = D_l$

at each point $p$ of $f(V)$.

**Invariant Submanifold of $\tilde{\psi}(3,2,1)$ Structure Manifold**

We call $V^m$ to be invariant submanifold of $M^n$ if the tangent space $T^p(f(V))$ of $f(V)$ is invariant by the linear mapping $\tilde{\psi}$ at each point $p$ of $f(V)$. Thus

$\tilde{\psi}BX = B\psi X$, for all $X \in \zeta_0^1(V)$,, and $\psi$ being a (1,1) tensor field in $V^m$. (2.1)

**Theorem (2.1)**

Let $\tilde{N}$ and $N$ be the Nijenhuis tensors determined by $\tilde{\psi}$ and $\psi$ in $M^n$ and $V^m$ respectively, then

$\tilde{N}(BX, BY) = BN(X, Y),$

for all $X, Y \in \zeta_0^1(V)$ (2.2)

**Proof**

We have, by using (1.2) and (2.1)

$$\begin{aligned}
(2.3)\quad \tilde{N}(BX, BY) &= [\tilde{\psi}BX, \tilde{\psi}BY] + \tilde{\psi}^2[BX, BY] \\
&\quad - \tilde{\psi}[\tilde{\psi}BX, BY] - \tilde{\psi}[BX, \tilde{\psi}BY] \\
&= [B\psi X, B\psi Y] + \tilde{\psi}^2 B[X, Y] \\
&\quad - \tilde{\psi}[B\psi X, BY] - \tilde{\psi}[BX, B\psi Y] \\
&= B[\psi X, \psi Y] + B\psi^2[X, Y] - \tilde{\psi}B[\psi X, Y] \\
&\quad - \tilde{\psi}B[X, \psi Y] \\
&= B\{[\psi X, \psi Y] + \psi^2[X, Y] - \psi[\psi X, Y] \\
&\quad - \psi[X, \psi Y]\} \\
&= B\,N(X, Y)
\end{aligned}$$

**Distribution $\tilde{M}$ Never Being Tangential To $f(V)$**

**Theorem (3.1)**

if the distribution $\tilde{M}$ is never tangential to $f(V)$, then $\tilde{m}(BX) = 0$ for all

$X \in \zeta_0^1(V)$ (3.1)

and the induced structure $\psi$ on $V^m$ satisfies

$\psi^2 + \psi + I = 0.$ Thus $\psi$ is (2,1,0) (3.2)

**Proof**

if possible $\tilde{m}(BX) \neq 0$. From (2.1) We get

$\tilde{\psi}^2 BX = B\psi^2 X;$ (3.3) from (1.7) and (3.3)

$$\begin{aligned}
\tilde{m}(BX) &= (I - \tilde{\psi}^3)BX \\
&= (I + \tilde{\psi} + \tilde{\psi}^2)BX \\
&= BX + B\psi X + B\psi^2 X
\end{aligned}$$

$$\tilde{m}(BX) = B(X + \psi X + \psi^2 X) \tag{3.4}$$

This relation shows that $\tilde{m}(BX)$ is tangential to $f(V)$ which contradicts the hypothesis. Thus $\tilde{m}(BX)$= 0. Using this result in (3.4) and remembering that $B$ is an isomorphism, We get

$$\psi^2 + \psi + I = 0 \tag{3.5}$$

**Theorem (3.2)**

Let $\tilde{M}$ be never tangential to $f(V)$, then

$$\tilde{N}_{\tilde{m}}\left(BX, BY\right)=0 \tag{3.6}$$

**Proof**

Wehave

$$\tilde{N}_{\tilde{m}}\left(BX, BY\right)=\left[\tilde{m}\, BX, \tilde{m}BY\right]+\tilde{m}^2\left[BX, BY\right]$$
$$-\tilde{m}\left[\tilde{m}BX, BY\right]-\tilde{m}\left[BX, \tilde{m}BY\right] \tag{3.7}$$

Using (1.2), (1.8) (c) and (3.1), we get (3.6).

**Theorem (3.3)**

Let $\tilde{M}$ be never tangential to $f\left(V\right)$, then

$$\tilde{N}_{\tilde{l}}\left(BX, BY\right)=0 \tag{3.8}$$

**Proof**

We have

$$\tilde{N}_{\tilde{l}}\left(BX, BY\right)=\left[\tilde{l}\, BX, \tilde{l}\, BY\right]+\tilde{l}^2\left[BX, BY\right]-\tilde{l}\left[\tilde{l}\, BX, BY\right]$$
$$-\tilde{l}\left[BX, \tilde{l}\, BY\right] \tag{3.9}$$

Using (1.2), (1.8) (a), (b) and (3.1) in (3.9); we get (3.8)

**Theoren (3.4)**

Let $\tilde{M}$ be never tangential to $f\left(V\right)$. Define

$$\tilde{H}\left(\tilde{X},\tilde{Y}\right)=\tilde{N}\left(\tilde{X},\tilde{Y}\right)-\tilde{N}\left(\tilde{m}\tilde{X},\tilde{Y}\right)-\tilde{N}\left(\tilde{X},\tilde{m}\tilde{Y}\right) \tag{3.10}$$
$$+\tilde{N}\left(\tilde{m}\tilde{X}, \tilde{m}\tilde{Y}\right)$$

For all $\tilde{X},\tilde{Y} \in \zeta_0^1\left(M\right)$, then

$$\tilde{H}\left(BX, BY\right)=BN\left(X,Y\right) \tag{3.11}$$

**Proof**

Using $\tilde{X}=BX,\ \tilde{Y}=BY$ and (2.2), (3.1) in (3.10) We get (3.11).

**Distribution $\tilde{M}$ Always Being Tangential To $f\left(V\right)$**

**Theorem (4.1)**

Let $\tilde{M}$ be always tangential to $f\left(V\right)$, then

(a) $\tilde{m}\left(BX\right)=Bm\,X$

(b) $\tilde{l}\left(BX\right)=Bl\,X$ (4.1)

**Proof**

from (3.4), We get (4.1) (a). Also

$$l=\psi^3 \tag{4.2}$$

$$lX=\psi^3\,X \tag{4.3}$$

$$BlX=B\psi^3\,X$$

Using (2.1) in (4.3)

$$BlX=\tilde{\psi}^3\,BX=\tilde{l}\left(BX\right), \tag{4.4}$$

which is (4.1) (b).

**Theorem (4.2)**

Let $\tilde{M}$ be always tangential to $f\left(V\right)$, then $l$ and $m$ satisfy

(a) $l+m=I$ (b) $lm=ml=0$ (c) $l^2=l$ (d) $m^2=m$. (4.5)

**Proof**

Using (1.8) and (4.1) We get the results.

**Theorem (4.3)**

If $\tilde{M}$ is always tangential to $f\left(V\right)$, then

$$\psi^3+\psi^2+\psi=0 \tag{4.6}$$

**Proof** From (2.1)

$$\tilde{\psi}^3\,BX=B\psi^3\,X \tag{4.7}$$

Using (1.6) in (4.7)

$$\left(-\tilde{\psi}^2-\tilde{\psi}\right)BX=B\psi^3\,X$$

$$-B\psi^2X-B\psi X=B\psi^3\,X$$

Or $\psi^3+\psi^2+\psi=0$ which is (4.6)

**Theorem (4.4)**

If $\tilde{M}$ Is always tangential to $f\left(V\right)$ then as in (3.10)

$$\tilde{H}\left(BX, BY\right)=BH\left(X,Y\right) \tag{4.7}$$

**Proof**

from (3.10) we get

$$\tilde{H}\left(BX, BY\right)=\tilde{N}\left(BX, BY\right)-\tilde{N}\left(\tilde{m}BX, BY\right)$$
$$-\tilde{N}\left(BX, \tilde{m}BY\right)+\tilde{N}\left(\tilde{m}BX, \tilde{m}BY\right)$$

Using (4.1) (a) and (2.2) in (4.9) we get (4.8).

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